Supplement to A Unified Analysis of Penalty-Based Collision Energies

ALVIN SHI and THEODORE KIM, Yale University, USA

ACM Reference Format:
Alvin Shi and Theodore Kim. 2023. Supplement to A Unified Analysis of Penalty-Based Collision Energies. *Proc. ACM Comput. Graph. Interact. Tech.* 6, 2 (August 2023), 5 pages. https://doi.org/10.1145/3606934

1 DERIVATION OF H

We provide here a detailed derivation of the signed length Hessian described in §4.2 of the main paper. In general, we derive symbolic expressions for the blocks of $\mathbf{H}_l$ by first taking the derivatives of l with respect to $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$, then apply simplifications as a result of the mutual orthogonality of $\mathbf{e}_{0\perp}$, $\mathbf{e}_{1\perp}$, $\mathbf{e}_{2\perp}$. The expressions become much cleaner after applying orthogonality assumptions, allowing for the subsequent eigenanalysis in the main paper.

1.1 Preliminaries

1.1.1 Differentiation. Let a and b be scalars and let $\mathbf{u}, \mathbf{v}, \mathbf{x}$ be vectors in $\mathbb{R}^n$. Let $\frac{\partial a}{\partial \mathbf{x}}$ be a column vector in $\mathbb{R}^n$ whose entries are $\frac{\partial a}{\partial x_i}$, and let $\frac{\partial \mathbf{u}}{\partial \mathbf{v}}$ be a matrix in $\mathbb{R}^{n\times n}$ such that $\left(\frac{\partial \mathbf{u}}{\partial \mathbf{v}}\right)_{ij} = \frac{\partial u_i}{\partial v_j}$. We note several generalized derivative rules:

$$\frac{\partial}{\partial \mathbf{x}}(a\mathbf{u}) = \mathbf{u}\left(\frac{\partial a}{\partial \mathbf{x}}\right)^\top + a\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \tag{1}$$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{u}\cdot\mathbf{v}) = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^\top \mathbf{u} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^\top \mathbf{v} \tag{2}$$

$$\frac{\partial}{\partial \mathbf{x}}(a(\mathbf{v}(\mathbf{x}))) = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^\top \frac{\partial a}{\partial \mathbf{v}} \tag{3}$$

1.1.2 Cross-Products. Cross-products frequently appear in several calculations. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{x}$ be independent vectors. We then have:

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{u}\times\mathbf{x}) = \mathbf{u}\times \tag{4}$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{x}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{x}) - \mathbf{x}(\mathbf{u}\cdot\mathbf{v}) \tag{5}$$

where $\mathbf{u}\times$ is the matrix such that $(\mathbf{u}\times)\mathbf{x} = \mathbf{u}\times\mathbf{x}$. Note that $(\mathbf{u}\times)^\top = -\mathbf{u}\times$.

Authors' address: Alvin Shi, alvin.shi@yale.edu; Theodore Kim, Yale University, New Haven, USA, theodore.kim@yale.edu.

2577-6193/2023/8-ART
https://doi.org/10.1145/3606934

1.1.3 Simplification from Orthogonality. Note that if $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$ are mutually orthogonal, we have several useful properties:

$$\mathbf{e}_i \times \mathbf{e}_j = \|\mathbf{e}_i\| \left\|\mathbf{e}_j\right\| \hat{\mathbf{e}}_k (\hat{\mathbf{e}}_k \cdot (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j)) \tag{6}$$

$$\left\|\mathbf{e}_i \times \mathbf{e}_j\right\| = \|\mathbf{e}_i\| \left\|\mathbf{e}_j\right\| \tag{7}$$

$$\hat{\mathbf{e}}_i \times = \left(\hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k)\right) \left(\hat{\mathbf{e}}_k \hat{\mathbf{e}}_j^\top - \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k^\top\right) \tag{8}$$

where $\hat{\mathbf{x}}$ is the unit vector $\mathbf{x}/\|\mathbf{x}\|$.

1.2 First Derivatives

To begin, recall that we are dealing with the signed length function

$$l = \frac{\mathbf{e}_2 \cdot (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}. \tag{9}$$

The first derivatives are:

$$\begin{aligned}
\frac{\partial l}{\partial \mathbf{e}_0} &= \frac{\partial}{\partial \mathbf{e}_0} \frac{\mathbf{e}_2 \cdot (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \qquad (10)\\
&= \left(\frac{\partial}{\partial \mathbf{e}_0}\left(\frac{\mathbf{e}_0 \times \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\right)\right)^\top \mathbf{e}_2 + \left(\frac{\partial \mathbf{e}_2}{\partial \mathbf{e}_0}\right)^\top \frac{\mathbf{e}_0 \times \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\\
&= \left((\mathbf{e}_0 \times \mathbf{e}_1)\left(\frac{\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3}\right)^\top - \frac{\mathbf{e}_1 \times}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\right)^\top \mathbf{e}_2\\
&= \frac{l\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\\
\frac{\partial l}{\partial \mathbf{e}_1} &= \left(\frac{\partial}{\partial \mathbf{e}_1}\left(\frac{\mathbf{e}_0 \times \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\right)\right)^\top \mathbf{e}_2 + \left(\frac{\partial \mathbf{e}_2}{\partial \mathbf{e}_1}\right)^\top \frac{\mathbf{e}_0 \times \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \qquad (11)\\
&= \left((\mathbf{e}_0 \times \mathbf{e}_1)\left(\frac{\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3}\right)^\top + \frac{\mathbf{e}_0 \times}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\right)^\top \mathbf{e}_2\\
&= \frac{l\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{\mathbf{e}_2 \times \mathbf{e}_0}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}\\
\frac{\partial l}{\partial \mathbf{e}_2} &= \frac{\mathbf{e}_0 \times \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}. \qquad (12)
\end{aligned}$$

Applying the orthogonality properties of $\mathbf{e}_{0\perp}, \mathbf{e}_{1\perp}, \mathbf{e}_{2\perp}$ yields

$$\frac{\partial l}{\partial \mathbf{e}_{0\perp}} = \mathbf{0}_3 \qquad \frac{\partial l}{\partial \mathbf{e}_{1\perp}} = \mathbf{0}_3 \qquad \frac{\partial l}{\partial \mathbf{e}_{2\perp}} = \frac{l\hat{\mathbf{e}}_{2\perp}}{\|\mathbf{e}_{2\perp}\|} \tag{13}$$

1.3 Second Derivatives

Analytic expressions for the blocks of $\mathbf{H}_l$ before applying orthogonality simplifications can become unwieldy, though there are some exceptions. First, in intermediate steps, we wrote $\frac{\partial l}{\partial \mathbf{e}_0}$ and $\frac{\partial l}{\partial \mathbf{e}_1}$ as matrices independent of $\mathbf{e}_2$ applied directly to $\mathbf{e}_2$, so taking another derivative with respect to $\mathbf{e}_2$ will result in those matrices. Second, since $\frac{\partial l}{\partial \mathbf{e}_2}$ has no dependence on $\mathbf{e}_2$, its derivative with respect to $\mathbf{e}_2$ is also zero.

The second derivatives are:

$$\frac{\partial^2 l}{\partial \mathbf{e}_2 \partial \mathbf{e}_0} = \left((\mathbf{e}_0 \times \mathbf{e}_1) \left(\frac{\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3} \right)^\top - \frac{\mathbf{e}_1 \times}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right)^\top \tag{14}$$

$$\frac{\partial^2 l}{\partial \mathbf{e}_2 \partial \mathbf{e}_1} = \left((\mathbf{e}_0 \times \mathbf{e}_1) \left(\frac{\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3} \right)^\top + \frac{\mathbf{e}_0 \times}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right)^\top \tag{15}$$

$$\frac{\partial^2 l}{\partial \mathbf{e}_2^2} = \mathbf{0}_{3\times 3} \tag{16}$$

Converting over to $\mathbf{e}_{0\perp}$, $\mathbf{e}_{1\perp}$, $\mathbf{e}_{2\perp}$ and applying properties from §1.1.3 leads to the final simplified forms:

$$\begin{aligned}
\frac{\partial^2 l}{\partial \mathbf{e}_{2\perp} \partial \mathbf{e}_{0\perp}} &= \frac{-l \mathbf{e}_{0\perp} \|\mathbf{e}_{1\perp}\|^2 \hat{\mathbf{e}}_{2\perp}^\top}{\|\mathbf{e}_{0\perp}\|^2 \|\mathbf{e}_{1\perp}\|^2 \|\mathbf{e}_{2\perp}\|} + \frac{\mathbf{e}_{1\perp} \times}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{1\perp}\|} && (17)\\
&= \frac{-l \hat{\mathbf{e}}_{0\perp} \hat{\mathbf{e}}_{2\perp}^\top}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{2\perp}\|} + \frac{\hat{\mathbf{e}}_{1\perp} \times}{\|\mathbf{e}_{0\perp}\|} && (18)\\
&= \frac{-l \hat{\mathbf{e}}_{2\perp} \hat{\mathbf{e}}_{0\perp}^\top}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{2\perp}\|} && (19)\\
\frac{\partial^2 l}{\partial \mathbf{e}_{2\perp} \partial \mathbf{e}_{1\perp}} &= \frac{-l \mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2 \hat{\mathbf{e}}_{2\perp}^\top}{\|\mathbf{e}_{0\perp}\|^2 \|\mathbf{e}_{1\perp}\|^2 \|\mathbf{e}_{2\perp}\|} - \frac{\mathbf{e}_{0\perp} \times}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{1\perp}\|} && (20)\\
&= \frac{-l \hat{\mathbf{e}}_{1\perp} \hat{\mathbf{e}}_{2\perp}^\top}{\|\mathbf{e}_{1\perp}\| \|\mathbf{e}_{2\perp}\|} - \frac{\hat{\mathbf{e}}_{0\perp} \times}{\|\mathbf{e}_{1\perp}\|} && (21)\\
&= \frac{-l \hat{\mathbf{e}}_{2\perp} \hat{\mathbf{e}}_{1\perp}^\top}{\|\mathbf{e}_{1\perp}\| \|\mathbf{e}_{2\perp}\|} && (22)
\end{aligned}$$

The remaining three blocks are slightly more involved. First, for the two diagonal blocks,

$$\begin{aligned}
\frac{\partial^2 l}{\partial \mathbf{e}_0^2} &= \frac{\partial}{\partial \mathbf{e}_0} \left(\frac{l\,(\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0))}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right) && (23)\\
&= (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)) \left(\frac{\partial l/\partial \mathbf{e}_0}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + l \frac{\partial}{\partial \mathbf{e}_0} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \right) \right)^\top && (24)\\
&\quad + \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \frac{\partial}{\partial \mathbf{e}_0} (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0))^\top\\
&\quad + (\mathbf{e}_1 \times \mathbf{e}_2) \frac{\partial}{\partial \mathbf{e}_0} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right)^\top\\
&= (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)) \left(\frac{\partial l/\partial \mathbf{e}_0}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{2l(\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0))}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^4} \right)^\top && (25)\\
&\quad + \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \left(\mathbf{e}_1 \mathbf{e}_1^\top - \|\mathbf{e}_1\|^2 \mathbf{I} \right)\\
&\quad + (\mathbf{e}_1 \times \mathbf{e}_2) \left(\frac{\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3} \right)^\top\\
\frac{\partial^2 l}{\partial \mathbf{e}_1^2} &= \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{l\,(\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1))}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{\mathbf{e}_2 \times \mathbf{e}_0}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right) && (26)
\end{aligned}$$

$$= (\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)) \left(\frac{\partial l/\partial \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + l \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \right) \right)^\top \tag{27}$$

$$+ \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \frac{\partial}{\partial \mathbf{e}_1} (\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1))$$

$$+ (\mathbf{e}_2 \times \mathbf{e}_0) \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right)^\top$$

$$= (\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)) \left(\frac{\partial l/\partial \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{2l(\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1))}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^4} \right)^\top \tag{28}$$

$$+ \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \left(\mathbf{e}_0 \mathbf{e}_0^\top - \|\mathbf{e}_0\|^2 \mathbf{I} \right)$$

$$+ (\mathbf{e}_2 \times \mathbf{e}_0) \left(\frac{\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3} \right)$$

Applying §1.1.3 leads to the compact expressions from the main paper:

$$\frac{\partial^2 l}{\partial \mathbf{e}_{0\perp}^2} = \mathbf{e}_{0\perp} \|\mathbf{e}_{1\perp}\|^2 \left(\frac{2l\mathbf{e}_{0\perp} \|\mathbf{e}_{1\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^4 \|\mathbf{e}_{1\perp}\|^4} \right)^\top \tag{29}$$

$$+ \frac{l \left(\mathbf{e}_{1\perp} \mathbf{e}_{1\perp}^\top - \|\mathbf{e}_{1\perp}\|^2 I \right)}{\|\mathbf{e}_{0\perp}\|^2 \|\mathbf{e}_{1\perp}\|^2}$$

$$- \|\mathbf{e}_{1\perp}\| l \hat{\mathbf{e}}_{0\perp} \left(\frac{\mathbf{e}_{0\perp} \|\mathbf{e}_{1\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^3 \|\mathbf{e}_{1\perp}\|^3} \right)^\top$$

$$= -\frac{l}{\|\mathbf{e}_{0\perp}\|^2} \left(\mathbf{I} - \hat{\mathbf{e}}_{1\perp} \hat{\mathbf{e}}_{1\perp}^\top - \hat{\mathbf{e}}_{0\perp} \hat{\mathbf{e}}_{0\perp}^\top \right) \tag{30}$$

$$\frac{\partial^2 l}{\partial \mathbf{e}_{1\perp}^2} = \mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2 \left(\frac{2l\mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^4 \|\mathbf{e}_{1\perp}\|^4} \right)^\top \tag{31}$$

$$+ \frac{l \left(\mathbf{e}_{0\perp} \mathbf{e}_{0\perp}^\top - \|\mathbf{e}_{0\perp}\|^2 \mathbf{I} \right)}{\|\mathbf{e}_{0\perp}\|^2 \|\mathbf{e}_{1\perp}\|^2}$$

$$- \|\mathbf{e}_{0\perp}\| l \hat{\mathbf{e}}_{1\perp} \left(\frac{\mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^3 \|\mathbf{e}_{1\perp}\|^3} \right)^\top$$

$$= \frac{-l}{\|\mathbf{e}_{1\perp}\|^2} \left(\mathbf{I} - \hat{\mathbf{e}}_{0\perp} \hat{\mathbf{e}}_{0\perp}^\top - \hat{\mathbf{e}}_{1\perp} \hat{\mathbf{e}}_{1\perp}^\top \right) \tag{32}$$

The final mixed term is as follows:

$$\frac{\partial^2 l}{\partial \mathbf{e}_1 \partial \mathbf{e}_0} = \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{l\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right) \tag{33}$$

$$= (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)) \left(\frac{\partial l/\partial \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + l \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \right) \right)^\top \tag{34}$$

$$+ \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \frac{\partial}{\partial \mathbf{e}_1} (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0))$$

$$+ (\mathbf{e}_1 \times \mathbf{e}_2) \frac{\partial}{\partial \mathbf{e}_1} \left(\frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \right)^\top + \frac{1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|} \frac{\partial}{\partial \mathbf{e}_1} (\mathbf{e}_1 \times \mathbf{e}_2)$$

$$= (\mathbf{e}_1 \times (\mathbf{e}_1 \times \mathbf{e}_0)) \left(\frac{\partial l / \partial \mathbf{e}_1}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} + \frac{2l(\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1))}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^4} \right)^\top \tag{35}$$
$$+ \frac{l}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^2} \left(\mathbf{e}_1 \mathbf{e}_0^\top + (\mathbf{e}_0 \cdot \mathbf{e}_1)\mathbf{I} - 2\mathbf{e}_0 \mathbf{e}_1^\top \right)$$
$$+ (\mathbf{e}_1 \times \mathbf{e}_2) \left(\frac{\mathbf{e}_0 \times (\mathbf{e}_0 \times \mathbf{e}_1)}{\|\mathbf{e}_0 \times \mathbf{e}_1\|^3} \right)^\top - \frac{\mathbf{e}_2 \times}{\|\mathbf{e}_0 \times \mathbf{e}_1\|}$$

Once again, applying §1.1.3 truncates the expression, resulting in the zero blocks reported in the main paper.

$$\frac{\partial^2 l}{\partial \mathbf{e}_{1\perp} \partial \mathbf{e}_{0\perp}} = \mathbf{e}_{0\perp} \|\mathbf{e}_{1\perp}\|^2 \left(\frac{2l\mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^4 \|\mathbf{e}_{1\perp}\|^4} \right)^\top \tag{36}$$
$$+ \frac{l}{\|\mathbf{e}_{0\perp}\|^2 \|\mathbf{e}_{1\perp}\|^2} (\mathbf{e}_{1\perp} \mathbf{e}_{0\perp}^\top - 2\mathbf{e}_{0\perp} \mathbf{e}_{1\perp}^\top)$$
$$- \|\mathbf{e}_{1\perp}\| \, l \hat{\mathbf{e}}_{0\perp} \left(\frac{\mathbf{e}_{1\perp} \|\mathbf{e}_{0\perp}\|^2}{\|\mathbf{e}_{0\perp}\|^3 \|\mathbf{e}_{1\perp}\|^3} \right)^\top - \frac{\mathbf{e}_{2\perp} \times}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{1\perp}\|}$$
$$= \frac{l(\hat{\mathbf{e}}_{1\perp} \hat{\mathbf{e}}_{0\perp}^\top - \hat{\mathbf{e}}_{0\perp} \hat{\mathbf{e}}_{1\perp}^\top) - \mathbf{e}_{2\perp} \times}{\|\mathbf{e}_{0\perp}\| \|\mathbf{e}_{1\perp}\|} = \mathbf{0}_{3\times 3} \tag{37}$$

With these blocks and the knowledge that Hessians must be symmetric, $\mathbf{H}_l$ is fully determined.

2 EIGENANLYSIS DETAILS

To derive eigenpairs of $\mathbf{H}_l$, note four observations:

$$\mathbf{H}_l \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{e}_{1\perp} \end{pmatrix} = \frac{-l}{\|\mathbf{e}_{2\perp}\|^2} \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \end{pmatrix} \qquad \mathbf{H}_l \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \end{pmatrix} = \frac{-l}{\|\mathbf{e}_{1\perp}\|^2} \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{e}_{2\perp} \\ \mathbf{e}_{1\perp} \end{pmatrix} \tag{38}$$

$$\mathbf{H}_l \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{e}_{0\perp} \end{pmatrix} = \frac{-l}{\|\mathbf{e}_{2\perp}\|^2} \begin{pmatrix} \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \\ \mathbf{0}_3 \end{pmatrix} \qquad \mathbf{H}_l \begin{pmatrix} \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \\ \mathbf{0}_3 \end{pmatrix} = \frac{-l}{\|\mathbf{e}_{0\perp}\|^2} \begin{pmatrix} \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \\ \mathbf{e}_{0\perp} \end{pmatrix} \tag{39}$$

Using the first two equations gives rise to a two-dimensional eigenproblem in the subspace $\left\{ a \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{e}_{1\perp} \end{pmatrix} + b \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \end{pmatrix} | a, b \in \mathbb{R} \right\}$ with the reduced matrix $\frac{-l}{\|\mathbf{e}_{1\perp}\|^2} \begin{pmatrix} 0 & 1 \\ \frac{\|\mathbf{e}_{1\perp}\|^2}{\|\mathbf{e}_{2\perp}\|^2} & 1 \end{pmatrix}$ acting on $\begin{pmatrix} a \\ b \end{pmatrix}$.

The second two equations give rise to a separate eigenproblem with $\left\{ a \begin{pmatrix} \mathbf{0}_3 \\ \mathbf{0}_3 \\ \mathbf{e}_{0\perp} \end{pmatrix} + b \begin{pmatrix} \mathbf{e}_{2\perp} \\ \mathbf{0}_3 \\ \mathbf{0}_3 \end{pmatrix} | a, b \in \mathbb{R} \right\}$ as the reduced space and $\frac{-l}{\|\mathbf{e}_{0\perp}\|^2} \begin{pmatrix} 0 & 1 \\ \frac{\|\mathbf{e}_{0\perp}\|^2}{\|\mathbf{e}_{2\perp}\|^2} & 1 \end{pmatrix}$ as the matrix.

Solving the resulting quadratics and expanding back into the full space, we obtain the eigensystem in the main paper.